The resilience effect for optimal execution problem (Theory and Application of Mathematical Decision Making under Uncertainty)

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1 Introduction

In the competitive market paradigm, it is assumed that security markets are perfectly elastic and all orders can be executed instantaneously. However in real markets, since institutional traders (large trader) usually submit orders considerable sizes, such traders thus give the influence to make the price change by their own dealings and have execution time lag. Such a price change that occurs at each trading period can be divided into two components, the temporary impact which represents the temporary cost of demanding liquidity and only affects an individual trade, and the permanent impact which influences the prices of all subsequent trades of an agent. These price changes may enable the large trader to manipulate the market. The act which manipulates a price intentionally and through managed to make profits actively spoils market public welfare, and is forbidden in many markets. With the appearance of electronic trading, this problem got more concerns in financial literacy. In this paper, under absence of price manipulation condition, we consider two types of price model about how to revert the previous price level for the buy trade. We show the properties of the optimal execution strategy of the large trader who reacts as a price maker and the time homogeneity of the parameters plays the important role for the market stability. We also show that the large trader leaves her holdings when she uses a price model of the gradual incorporation of a piece of trading information into the price.

The paper is organized as follows. Section 2 presents two price models and discuss the price impact and the price manipulation briefly. Section 3 presents the optimal execution strategy with dynamic programming algorithm for two price models. Section 4 presents the properties of optimal execution volume with time-homogeneous parameters and illustrates numerical examples. Section 5 concludes the paper.

2 Market Model and Price Manipulation

In this section, we explain two discrete time price models called the permanent price model and the transient price model. The reason we use these price models is that these price models satisfy the absence of price manipulation condition under certain conditions. So we also introduce the definition of price manipulation.

Consider a discrete time dynamic execution problem in which a risk averse large trader plans to purchase a large amount of $\bar{Q}$ shares of a risky asset over equidistant time interval $[0, T]$. In general, the large trader breaks the block trade into small slices since the demand of the large trader may be beyond the supply of the order book, or the large amount of order may change a price. She is also permitted to submit sell orders over trading time and assume the symmetric impact, that is, a price impact of selling and buying is equal. The superscript of each variable denoting $i = pe$ or tr represents for using the permanent price model or the transient price model

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respectively. Suppose that p_t^i is the price of a single risky asset at time t, q_t is the large trader's execution volume. It is the buy trade if $q_t > 0$, on the other hand if $q_t < 0$, the sell trade. Q_t is the number of shares which the large trader remains to purchase, if $Q_t > 0$(or liquidate, if $Q_t < 0$). That is,

$$Q_{t+1} = Q_t - q_t. \tag{2.1}$$

Moreover, w_t^i is a wealth at time t. The large trader submits a large amount of her market order q_t at time t just after she has recognized the price p_t^i at that time. Although the order is executed immediately, the execution price may not be equal to p_t^i. The executed price $\hat{p}_t^i$ will be instantly lifted upward from p_t^i because of the temporary imbalance of supply and demand. Assume that λ_t denote to the price change per share (called price impact), the dynamics of w_t^i and $\hat{p}_t^i$ are,

$$w_{t+1}^i = w_t^i - \hat{p}_t^i q_t, \tag{2.2}$$

$$\hat{p}_t^i = p_t^i + \lambda_t q_t. \tag{2.3}$$

The lifted price (execution price) by the large order recurs to previous price level to a certain extent since temporary imbalance of supply and demand on the order book moves to new equilibrium with progress of the time. We introduce following two price models regarding the way how to revert the execution price. One is the permanent price model which was mainly introduced by [7] and more previously [10], the other is the transient price model considered by [5] and [11].

2.1 Price dynamics

Permanent price model: In the permanent price model, the execution price is diminished instantly to the permanent impact level and the expected price is maintained until the next trading time. That is,

$$p_{t+1}^{pe} = \alpha_t p_t^{pe} + (1-\alpha_t)\hat{p}_t^{pe} + \epsilon_{t+1}. \tag{2.4}$$

Using equation (2.3) and (2.4),

$$p_{t+1}^{pe} = p_t^{pe} + (1-\alpha_t)\lambda_t + \epsilon_{t+1}, \tag{2.5}$$

where α_t represents the deterministic reversion rate of price and follows $0 \le \alpha_t \le 1$. ϵ_{t+1} represents the public news effect to the fundamental price between time t and $t+1$ and is recognized by the large trader at time $t+1$. Further, $\{\epsilon_t\}_t$ are *i.i.d* random variables on a probability space $(\Omega, \mathcal{F}, P)$ as follows,

$$\epsilon_t \sim N(0, \sigma_\epsilon^2). \tag{2.6}$$

All information available to the large trader before her trading at time t are,

$$\mathcal{F}_t := \sigma\{(\epsilon_{s+1}) : s = 1, \ldots, t-1\}. \tag{2.7}$$

In the permanent price model, the price impact, the temporary impact and the permanent impact are represented respectively λ_t, $(1-\alpha_t)\lambda_t$ and $\alpha_t\lambda_t$. The temporary impact means the temporal imbalance of supply and demand and the permanent impact is the new price update by the information of the trading at time t. In the permanent price model, since the trading information at time t incorporate into the price immediately, then $(temporary\ impact) + (permanent\ impact) = (price\ impact)$. As far as the time intervals are fixed equally, this price model is same as that of [2] price model (linear reversion rate model).

Transient price model: In the transient price model, on the other hand, it is the same as the permanent price model until the submitted order is executed. However the price recurrence is not immediate but gradual to permanent level. We set the time independent rate ρ as the resilience speed. Then,

$$p_t^{tr} = p_t^0 + \sum_{k=1}^{t-1} \lambda_k e^{-\rho(t-k)} q_k, \tag{2.8}$$

where p^0 denotes the fundamental price and $p_{t+1}^0 - p_t^0 =: \epsilon_{t+1}$, the same as (2.6) and (2.7). Furthermore, by the equation(2.8),

$$p_{t+1}^{tr} = p_t^{tr} + \lambda_t e^{-\rho} q_t - S_t + \epsilon_{t+1}. \tag{2.9}$$

Here, we define S as,

$$S_t := e^{-\rho t}\left(1 - e^{-\rho}\right) \sum_{k=1}^{t-1} \lambda_k e^{-\rho(t-k)} q_k = l_{t-1} q_{t-1} + e^{-\rho} S_{t-1}, \tag{2.10}$$

where,

$$l_t := \lambda_t (1 - e^{-\rho}) e^{-\rho}. \tag{2.11}$$

In this transient price model, price impact and transient impact are λ_t and $\lambda_t e^{-\rho(t-k)}$. On the other hand, temporary and permanent impact are both 0. The economic interpretation of S_t is the difference of the cumulative transient impact traded from time 1 to $t-1$ between viewed at the time t and at the time $t+1$. Since the price reverts to the permanent level over and over (in this case price is down), then $S_t \geq 0$.

The main difference between these two price models is whether the effect of the present execution is completely incorporated in the price immediately or not. In the transient price model, since the price after the present execution fall down gradually to the permanent level (in this case 0), the effect of the present execution is partially incorporated in the price at the following trading time, and completely incorporated with progress of the time.

2.2 Price manipulation

Here, we introduce the two definitions of price manipulation.

Definition 1 (Price manipulation [6]) *A round trip trade is an execution strategy $\{q_t\}_t$ such that, $\sum_{t=1}^T q_t = 0$. A price manipulation strategy is a round trip trade such that, $E\left[\sum_{t=1}^T \hat{p}_t q_t\right] < 0$.*

In [6], they showed if the permanent impact is linear in terms of the execution volume, then the market is absent of price manipulation in the risk neutral sense. Within the time-homogeneous reversion rate framework, our permanent price model satisfies the condition of absence of price manipulation. Our control for the risk averse large trader describes that when we apply the round trip trade, 0 trade is always optimal.

Definition 2 (Transaction-triggered price manipulation [1]) *If the expected execution costs of a buy program can be decreased by intermediate sell trade, the price model admits transaction-triggered price manipulation. That is, there exists $Q_1, T > 0$ and a corresponding execution* *strategy $\tilde{q}$ for which under a monotone execution strategy q,*

$$E\left[C_T\left(\tilde{q}\right)\right] < \min E\left[C_T\left(q\right)\right]. \tag{2.12}$$

Our transient price model with exponential resilience satisfies the condition of absence of transaction-triggered price manipulation.

Definition 2 is a stronger condition of the price manipulation than that of Definition 1. That is to say, even if the price model satisfies the ordinal price manipulation, it may not be satisfy the transaction-triggered price manipulation, such as buy and sell oscillation trades.

3 Optimal execution

In this subsection, we show the optimal execution strategy exists in the static class by deriving the explicit solution with the dynamic programming approach. Suppose that a risk-averse large trader with CARA(Constant Absolute Risk Aversion) type utility of which the risk aversion coefficient is R submits large amount of market order at the equally time distance over the maturity T. We consider the problem of the dynamic execution strategy that maximizes her expected utility from her terminal wealth. Here we show optimal execution strategy based mainly on the transient price model. For the permanent price model, we only provide the result since it is more simple calculation ($S_t = 0$).
We define her expected utility under the trading strategy π at time t as,

$$V_t^{\pi} := E_t^{\pi}\left[-\exp\{-Rw_{T+1}^{tr}\}1_{\{Q_{T+1}=0\}} + (-\infty)1_{\{Q_{T+1}\neq 0\}}\right], \quad t = 1, \cdots, T, \tag{3.1}$$

where $1_{\{\bullet\}}$ is indicator function and the right hand side of the equation (3.1) represents that it is optimal for the large trader to execute whole her holdings at maturity T. Moreover we define the optimal value function as,

$$V_t := \operatorname*{esssup}_{\pi} V_t^{\pi}, \quad t = 1, \cdots, T. \tag{3.2}$$

The subscript t of the expectation represents the condition under all information available to the large trader at time t.

Because of the Markov property of the dynamics and path independency of her utility at the final period, V_t is a function of $(w_t^{tr}, p_t^{tr}, Q_t, S_t)$, and by the principle of optimality, the optimality equation (Bellman equation) becomes as,

$$V_t\left(w_t^{tr}, p_t^{tr}, Q_t, S_t\right) = \sup_{q_t \in R} E\left[V_{t+1}(w_{t+1}^{tr}, p_{t+1}^{tr}, Q_{t+1}, S_{t+1}) \mid w_t^{tr}, p_t^{tr}, Q_t, S_t\right]. \tag{3.3}$$

We solve the sequence of optimal execution volume which achieves V_1 from the final period T by backward induction in t. In the following we only show the optimal execution strategies without considering the uncertainty on the order book. The optimal strategy for more complicated price models, the detailed derivation and the proof of Theorem 1 and Corollary 1, refer to [8] and [9].

Transient price model:

Theorem 1 *When we use the transient price model, the optimal execution volume of large trader at time t denoting q_t^* is represented as the function of the remaining execution volume Q_t*

and the cumulative effect of past executions S_t at the time. Then the optimal execution volume and the corresponding optimal value function are,

$$q_t^* = \frac{D_t Q_t - L_t S_t}{2C_t} \quad (= \beta_t Q_t - \gamma_t S_t), \tag{3.4}$$

where,

$$\begin{cases} C_t := l_t e^{\rho} + \frac{R\sigma_\epsilon^2}{2} + A_{t+1} - B_{t+1} l_t - K_{t+1} l_t^2 \\ D_t := -\lambda_t e^{-\rho} + R\sigma_\epsilon^2 + 2A_{t+1} - B_{t+1} l_t \\ L_t := 1 - B_{t+1} e^{-\rho} - 2K_{t+1} l_t e^{-\rho}, \end{cases} \tag{3.5}$$

and the value function is,

$$V_t\left(w_t^{tr}, p_t^{tr}, Q_t, S_t\right) = -\exp\{-R(w_t^{tr} - p_t^{tr} Q_t - A_t Q_t^2 - B_t S_t Q_t + K_t S_t^2)\}, \tag{3.6}$$

where,

$$\begin{cases} A_t := A_{t+1} + \frac{R\sigma_\epsilon^2}{2} - \frac{D_t^2}{4C_t} \\ B_t := B_{t+1} e^{-\rho} - 1 + \frac{D_t L_t}{2C_t} \\ K_t := K_{t+1} e^{-2\rho} + \frac{L_t^2}{4C_t}. \end{cases} \tag{3.7}$$

Then a deterministic execution strategy becomes optimal.

Permanent price model:

Corollary 1 *When we use the permanent price model, the optimal execution volume at time t denoting $q_t^{'*}$ is represented as an affine function of the remaining execution volume Q_t at the time. Then the optimal execution volume and the optimal value function are,*

$$q_t^{'*} = \frac{D_t' Q_t}{2C_t'} \quad (= \beta_t' Q_t), \tag{3.8}$$

where,

$$\begin{cases} C_t' := \alpha_t \lambda_t + \frac{R\sigma_\epsilon^2}{2} + A_{t+1}' \\ D_t' := -(1-\alpha_t)\lambda_t + R\sigma_\epsilon^2 + 2A_{t+1}', \end{cases} \tag{3.9}$$

and the value function is,

$$V_t\left(w_t^{pr}, p_t^{pr}, Q_t\right) = -\exp\{-R(w_t^{pr} - p_t^{pr} Q_t - A_t' Q_t^2)\}, \tag{3.10}$$

where,

$$A_t' := A_{t+1}' + \frac{R\sigma_\epsilon^2}{2} - \frac{D_t'^2}{4C_t'}. \tag{3.11}$$

The optimal solutions for the transient price model consists of two components, β and γ. β contributes directly to the optimal solution, on the other hand, γ contributes secondarily. If external factors are incorporated into the both price models, additional terms are added besides β', or β and γ ([8] and [9]).

Remark 1 *For both price model, Q_t can be expressed in β, γ, S, β' and $\bar{Q} = Q_1$. Therefore, by Equation (2.10), Q_t can be controlled determinately. For the transient price model,*

$$Q_t = \prod_{i=1}^{t-1}(1-\beta_i)\bar{Q} + \sum_{k=3}^{t}\left[\prod_{i=k}^{t}(1-\beta_i)\right]\gamma_{k-1}S_{k-1} + \gamma_t S_t, \tag{3.12}$$

and for the permanent price model Q',

$$Q'_t = \prod_{i=1}^{t-1}(1-\beta'_i)\bar{Q}. \tag{3.13}$$

4 Properties and comparative statics

In this section, we provide the properties of the optimal execution strategies for the price models previously shown. For simplicity, we assume the time homogeneity of price impact $\lambda_t = \lambda$, reversion rate in permanent price model $\alpha_t = \alpha$ and resilience speed in the transient price model $\rho_t = \rho$ in this section. For each parameters R, σ^2, λ, ρ and α, we also illustrate intraday execution strategies for the large trader. We divide intraday into 13 periods (30 minutes) based on NYSE and assume that the large trader must purchase 100,000 shares of risky asset within 13 periods ([3]). For each figure, only one parameter is transformed and others are fixed. For the fixed parameter, we set $R = 0.001$, $\lambda = 0.0005$, $\sigma^2 = 0.01$, $\rho = 0.01$ and $\alpha = 0.6$.

Suppose R_a and R_b are the risk aversion coefficient of the large trader a and b and $Q(\cdot)$ represents a function of remaining volume to purchase in the parenthesis. Firstly, we provide several properties for the differences of risk aversion R and presume the following conjecture.

Conjecture: *Suppose that q_t^* or $q_t'^*$ $(t = 1, \ldots, T)$ are the optimal execution volumes for the transient or the permanent price model respectively. Under the assumption of time homogeneity, the optimal execution strategies for both price models are convex in time.*
This property will be true in numerical experiment (see Figure 1), but the analytic proof is ongoing research. In the transient price model, if we admit that this conjecture is true, the following two propositions can be proved easily.

Proposition 1 *If the large trader is risk-neutral, then the optimal strategy is time-reversible, i.e. $q_t^* = q_{T-t+1}^*$.*

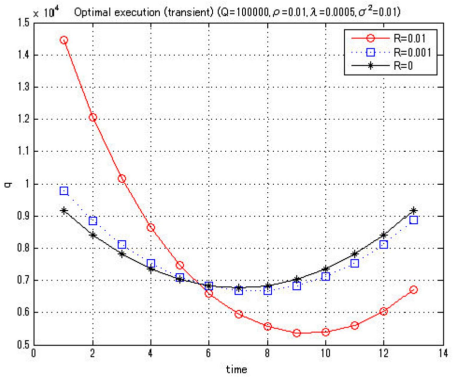
Proposition 2 *Under the assumption of time homogeneity, if $\rho \neq 0$, the optimal execution strategy q_t^* satisfies,*

$$q_1^* \geq q_2^* \quad \text{and} \quad q_{T-1}^* \leq q_T^*. \tag{4.1}$$

The proof of Proposition 1 and Proposition 2 are similar to the Proposition 4 and Proposition 6 in [1] because of the monotonicity of exponential function in the utility function and the convexity of V in terms of q. Moreover if $\rho = 0$, the shape of the optimal execution strategy is same as that for the permanent price model (following Lemma 1), because of the full incorporation of trading information into the price at each time.

Lemma 1 *Under the assumption of time homogeneity, the optimal execution strategy for the permanent price model decreases monotonously in time. That is,*

$$q_1'^* \geq q_2'^* \geq \ldots \geq q_T'^*. \tag{4.2}$$



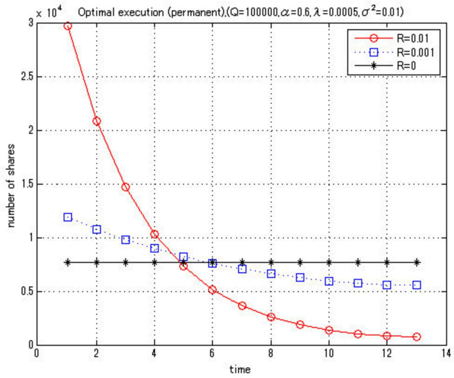


Figure 1: Optimal execution strategies for transient price model (the left half) and permanent price model (the right half) with various risk aversion coefficients.

Proposition 3 *For the permanent price model, if the large trader is risk-neutral then the TWAP strategy is optimal. That is, the optimal execution strategy q_t^* satisfies,*

$$q_1^{'*} = q_2^{'*} = \ldots = q_T^{'*}. \tag{4.3}$$

Proposition 4 (Risk Aversion effect) *For two price models, the more risk averse the large trader is, the earlier she executes. That is, for all t, if $R_a \geq R_b$, then*

$$(0 \leq)\ Q_t^{(')}(R_a) \leq Q_t^{(')}(R_b). \tag{4.4}$$

Proposition 3 indicates that if $R \downarrow 0$, then the optimal execution strategy with price impact is equal to the naive strategy [4]. For the detailed proof of Lemma 1, refer to [8] and for that of Proposition 3 and 4, refer to [9] which can be proved without Conjecture.

These propositions and lemma show that for the permanent price model, the optimization problem is the simple trade-off between impact risk by accelerating the execution and price fluctuation (volatility) risk by delaying the execution, on the other hand for the transient price model, the large trader considers the expectation of the price reversion besides these trade-off. Therefore, since $S_t \geq 0$, the large trader leaves her holdings until later time.

The left half of Figure 1 illustrates these properties for transient price model and the right is for permanent price model. Figure 1 illustrates the optimal execution strategies of the large trader having various risk averse coefficients ($R = 0, 0.001, 0.01$). These two figures show the convexity of the optimal execution strategies for both two price models in time and if $R = 0$, it is optimal for transient price model to execute U-shape in time and for permanent price model to execute equally in time. Next, we analyze the effect of market impact λ and market volatility σ^2.

Proposition 5 *If the market is fully liquid then it is optimal for risk-averse large trader to execute at the initial time.*

Proof. For the transient price model, if $\lambda \to 0$, then $\beta_t \to 1$ and $S_t \to 0$ for all t. And for the permanent price model, if $\lambda \to 0$, then $\beta_t \to 0$ for all t. □

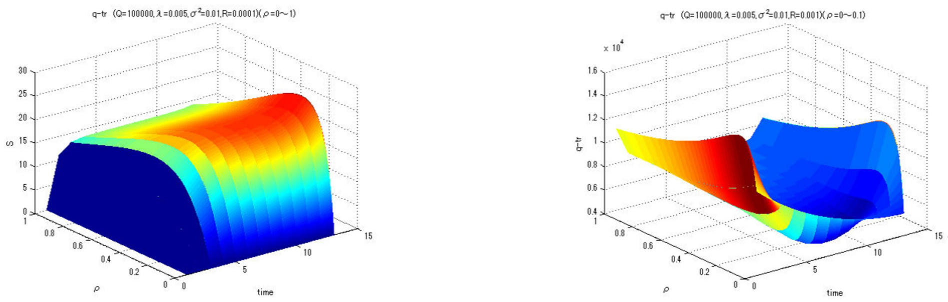


Figure 2: The relationship between S (the left half) and optimal execution strategy (the right half) for the transient price model with various ρ.

Remark 2 *For two price models, the more the market is volatile, the earlier the large trader executes. That is, for all t, if $\sigma_a^2 \geq \sigma_b^2$, then*

$$(0 \leq)\ Q_t^{(')}(\sigma_a^2) \leq Q_t^{(')}(\sigma_b^2). \tag{4.5}$$

From Equation (3.5), (3.7), (3.9) and (3.11), the proof of this remark is the same as R (Proposition 4) because the ratio of change is the same as R.

The optimal execution strategy for the transient price model when the value of ρ is taken from 0 to 1 is illustrated in Figure 2. This figure shows that the shape of resilience (ρ or S) also greatly influence the optimal execution strategy. Finally, we provide a sufficient condition for the absence of transaction-triggered price manipulation for the permanent price model.

Theorem 2 *Under the assumption of time homogeneity, the optimal execution strategy for the permanent price model does not admit the transaction-triggered price manipulation.*

Using Lemma 1, we can easily show that $\beta_t^{'} \geq 0$ straightforward calculation. So we omit the detailed proof. Figure 3 illustrates that optimal execution for the permanent price model with time-inhomogeneous α_t (for example $\alpha_t = 1 - e^{-\rho t}$) may violate the absence of transaction-triggered price manipulation when $\rho = 0.1, 0.8$.

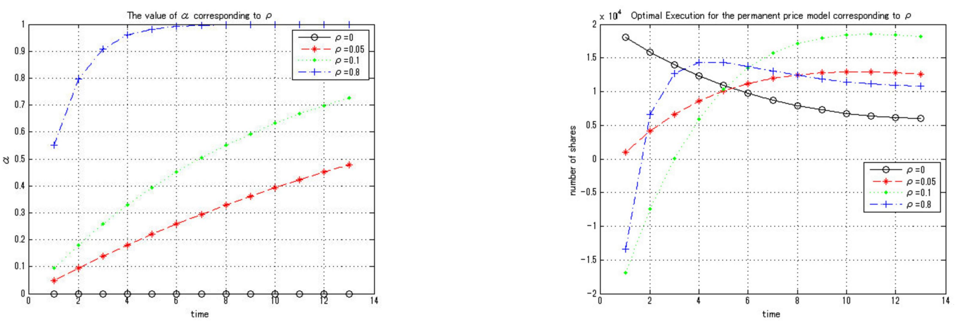


Figure 3: The corresponding value of α when $\alpha_t = 1 - e^{-\rho t}$ and $\rho = 0, 0.05, 0.1, 0.8$ (the left half) and optimal execution strategy (the right half) for the transient price model corresponding to these values of ρ or α_t.

5 Conclusion

This paper investigated properties of optimal execution volume with price impact derived from dynamic programming algorithm in the discrete time setting by calculating directly from the maturity time. Under some assumptions that the large trader with CARA type utility and normality of random variables of public news effect to the price, the optimal execution strategy exists in the static class. Furthermore, we showed the properties of the execution performance for two price models focusing on the resilience and confirmed that the time homogeneity of the parameter greatly affects the price manipulation. More practical models such as the intraday liquidity effect etc. and the detailed property such as a convexity of the optimal execution strategy are remaining for future works.

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